Topological applications of Wadge theory III

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A sufficient condition for reasonability

Lemma (Step 3)

Let $\mathbf{\Gamma}$ be a Wadge class in 2^ω that is closed under intersections with $\mathbf{\Pi}^0_2$ sets and unions with $\mathbf{\Sigma}^0_2$ sets. Then $\mathbf{\Gamma}$ is reasonably closed.

Proof.

Pick $A \in \mathbf{\Gamma}$. We need to show that $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$.
By Van Wesep's Theorem, fix $D \subseteq \mathcal{P}(\omega)$ such that $\mathbf{\Gamma} = \mathbf{\Gamma}_D(2^\omega)$.

Set $Z = 2^\omega \setminus (Q_0 \cup Q_1)$, and notice that $\phi^{-1}[A] \in \mathbf{\Gamma}_D(Z)$ by the Relativization Lemma. Therefore, again by the Relativization Lemma, there exists $B \in \mathbf{\Gamma}_D(2^\omega) = \mathbf{\Gamma}$ such that $B \cap Z = \phi^{-1}[A]$.

Since $Z \in \mathbf{\Pi}^0_2(2^\omega)$, it follows from our assumptions that $\phi^{-1}[A] \in \mathbf{\Gamma}$, hence $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$. $\square$

In particular, it follows that both $\mathbf{\Sigma}^0_\xi(2^\omega)$ and $\mathbf{\Pi}^0_\xi(2^\omega)$ are reasonably closed whenever $3 \leq \xi < \omega_1$. (But we will need to be much more sophisticated than that!)

Three steps to reasonability

$$\begin{array}{c}
\boldsymbol{\Gamma} = [X] \text{ for some homogeneous } X \subseteq 2^\omega \text{ such that } X \notin \Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_2)) \\
\downarrow \\
\boldsymbol{\Gamma} \text{ is a good Wadge class} \\
\downarrow \\
\boldsymbol{\Gamma} \text{ is closed under } \cap\, \boldsymbol{\Pi}^0_2 \text{ and } \cup\, \boldsymbol{\Sigma}^0_2 \\
\downarrow \\
\boldsymbol{\Gamma} \text{ is reasonably closed}
\end{array}$$

The notion of level

From now on, $\xi < \omega_1$ and $\boldsymbol{\Gamma}$, $\boldsymbol{\Lambda}$ are Wadge classes in Z.

Definition (Louveau, Saint-Raymond, 1988)

- $\ell(\boldsymbol{\Gamma}) \geq \xi$ if $\mathrm{PU}_\xi(\boldsymbol{\Gamma}) = \boldsymbol{\Gamma}$
- $\ell(\boldsymbol{\Gamma}) = \xi$ if $\ell(\boldsymbol{\Gamma}) \geq \xi$ and $\ell(\boldsymbol{\Gamma}) \not\geq \xi + 1$
- $\ell(\boldsymbol{\Gamma}) = \omega_1$ if $\ell(\boldsymbol{\Gamma}) \geq \xi$ for every $\xi < \omega_1$

We refer to $\ell(\boldsymbol{\Gamma})$ as the *level* of $\boldsymbol{\Gamma}$.

Examples:

- $\ell(\boldsymbol{\Gamma}) \geq 0$ for every $\boldsymbol{\Gamma}$
- $\ell(\{\varnothing\}) = \ell(\{Z\}) = \omega_1$
- $\ell(\boldsymbol{\Sigma}^1_n) = \ell(\boldsymbol{\Pi}^1_n) = \omega_1$ for every $n \in \omega$
- $\ell(\boldsymbol{\Sigma}^0_{1+\xi}) = \ell(\boldsymbol{\Pi}^0_{1+\xi}) = \xi$

Notice that if $\ell(\boldsymbol{\Gamma}) \geq \xi$ then $\boldsymbol{\Gamma}$ is closed under $\cap \boldsymbol{\Delta}^0_{1+\xi}$.

The expansion theorem

Definition (Wadge, 1984)

$$\mathbf{\Gamma}^{(\xi)} = \{f^{-1}[A] : A \in \mathbf{\Gamma} \text{ and } f : Z \longrightarrow Z \text{ is } \mathbf{\Sigma}^0_{1+\xi}\text{-measurable}\}$$

We will refer to $\mathbf{\Gamma}^{(\xi)}$ as an *expansion* of $\mathbf{\Gamma}$. To see what happens with regard to Hausdorff operations, it can be shown that

$$\mathbf{\Gamma}_D(Z)^{(\xi)} = \{\mathcal{H}_D(A_0, A_1, \ldots) : A_0, A_1, \ldots \in \mathbf{\Sigma}^0_{1+\xi}(Z)\}$$

Theorem (Louveau)

Assume that $\mathbf{\Gamma}$ is non-selfdual. Then the following conditions are equivalent:

- $\ell(\mathbf{\Gamma}) \geq \xi$
- *There exists a non-selfdual $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma}$*

Good Wadge classes

Definition

We will say that $\mathbf{\Gamma}$ is *good* if the following are satisfied:

- $\mathbf{\Gamma}$ is non-selfdual
- $\Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)) \subseteq \mathbf{\Gamma}$
- $\ell(\mathbf{\Gamma}) \geq 1$

Lemma (Step 2)

If $\mathbf{\Gamma}$ is good then $\mathbf{\Gamma}$ is closed under $\cap \mathbf{\Pi}^0_2$ and $\cup \mathbf{\Sigma}^0_2$.

Proof.

Andretta, Hjorth and Neeman proved that if $\Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_1)) \subseteq \mathbf{\Lambda}$ then $\mathbf{\Lambda}$ is closed under $\cap \mathbf{\Pi}^0_1$ and $\cup \mathbf{\Sigma}^0_1$. Since $\ell(\mathbf{\Gamma}) \geq 1$ there exists $\mathbf{\Lambda}$ such that $\mathbf{\Lambda}^{(1)} = \mathbf{\Gamma}$. Apply the above mentioned result to $\mathbf{\Lambda}$, then transfer it to $\mathbf{\Gamma}$ using expansions. $\square$

The proof of Step 1

Let $X \subseteq 2^\omega$ be dense and homogeneous, with $X \notin \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$. We need to show that $[X]$ is a good Wadge class.

Fix a minimal non-selfdual $\mathbf{\Gamma}$ such that there exists a non-empty $U \in \mathbf{\Sigma}^0_1(2^\omega)$ such that $X \cap U \in \mathbf{\Gamma}$ or $X \cap U \in \check{\mathbf{\Gamma}}$. Assume that $X \cap U \in \mathbf{\Gamma}$. First we will show that $\mathbf{\Gamma}$ is good, then that $[X] = \mathbf{\Gamma}$.

Assume, in order to get a contradiction, that $X \cap U \in \Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$. Notice that $\mathcal{U} = \{h[X \cap U] : h \text{ is a homeomorphism of } X\}$ is a cover of X because X is homogeneous and dense in 2^ω.

Furthermore, since $\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$ is a good Wadge class, the following lemma shows that each element of $\mathcal{U}$ belongs to it:

Lemma (Good Wadge classes are "topological")

Let $\mathbf{\Gamma}$ be a good Wadge class in Z. If $A \in \mathbf{\Gamma}$ and $B \approx A$ then $B \in \mathbf{\Gamma}$.

Using a countable subcover of $\mathcal{U}$, write X as a partitioned union of sets in $\mathrm{D}_\omega(\mathbf{\Sigma}^0_2)$, where the elements of the partition are $\mathbf{\Delta}^0_2$.

Since $\ell(\mathsf{D}_\omega(\boldsymbol{\Sigma}^0_2)) \geq 1$, it follows that $X \in \mathsf{D}_\omega(\boldsymbol{\Sigma}^0_2)$. A similar argument shows that $X \in \check{\mathsf{D}}_\omega(\boldsymbol{\Sigma}^0_2)$. This contradicts our assumptions, so $X \cap U \notin \Delta(\mathsf{D}_\omega(\boldsymbol{\Sigma}^0_2))$.

It remains to show that $\ell(\boldsymbol{\Gamma}) \geq 1$. Assume, in order to get a contradiction, that $\ell(\boldsymbol{\Gamma}) = 0$. Then, applying the following with $Z = U$ will contradict the minimality of $\boldsymbol{\Gamma}$:

Lemma

Assume that $\boldsymbol{\Gamma}$ is non-selfdual and that $\ell(\boldsymbol{\Gamma}) = 0$. Let $X \in \boldsymbol{\Gamma}$ be codense in Z. Then there exist a non-empty $V \in \boldsymbol{\Delta}^0_1(Z)$ and a non-selfdual $\boldsymbol{\Lambda}$ such that $\boldsymbol{\Lambda} \subsetneq \boldsymbol{\Gamma}$ and $X \cap V \in \boldsymbol{\Lambda}$.

Now that we know that $\boldsymbol{\Gamma}$ is a good Wadge class, since $X \cap U \in \boldsymbol{\Gamma}$, we can apply the same homogeneity argument as above to see that $X \in \boldsymbol{\Gamma}$, so $[X] \subseteq \boldsymbol{\Gamma}$. It remains to show that $[X] \subsetneq \boldsymbol{\Gamma}$ is impossible.

If X is non-selfdual, this would directly contradict minimality of $\boldsymbol{\Gamma}$. Otherwise, minimality would be contradicted after applying the analysis of the selfdual sets.

Finishing the proof

Let X be a zero-dimensional homogeneous space that is not locally compact. Without loss of generality, assume that X is a dense subspace of 2^ω. If $X \in \Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_2))$, then X is strongly homogeneous by van Engelen's results. Therefore, we can also assume without loss of generality that $X \notin \Delta(\mathrm{D}_\omega(\boldsymbol{\Sigma}^0_2))$.

Fix $s \in 2^{<\omega}$, and let $Y = X \cap [s]$. As in the proof of Step 1, using also the Relativization Lemma, one can show that X and Y are everywhere properly $\boldsymbol{\Gamma} = [X]$ (in 2^ω and $[s] \approx 2^\omega$ respectively).

Since X is homogeneous, either X is meager or it is Baire (hence comeager in 2^ω by AD). The same will be true of Y.

Hence $Y \approx X$ by Steel's theorem. The following result concludes the proof that X is strongly homogeneous:

Theorem (Terada, 1993)

Let X be a space. Assume that X has a base $\mathcal{B} \subseteq \boldsymbol{\Delta}^0_1(X)$ such that $U \approx X$ for every $U \in \mathcal{B}$. Then X is strongly homogeneous.

Open questions

As we have seen, for spaces of complexity higher than $\Delta(\mathrm{D}_\omega(\mathbf{\Sigma}^0_2))$, Baire category and Wadge class are sufficient to uniquely identify a homogeneous zero-dimensional space. This is the "uniqueness" part of the classification. But the "existence" part is still open:

Question

For exactly which good Wadge classes $\mathbf{\Gamma}$ is there a homogeneous X such that $\mathbf{\Gamma} = [X]$? For which ones is there a meager such X? For which ones is there a Baire such X?

Does the usual pattern of results under AD hold?

Question

Assuming $\mathrm{V} = \mathrm{L}$, is it possible to construct a zero-dimensional $\mathbf{\Pi}^1_1$ or $\mathbf{\Sigma}^1_1$ space that is homogeneous, not locally compact, and not strongly homogeneous?

What happens below $\Delta(\mathsf{D}_\omega(\boldsymbol{\Sigma}^0_2))$?

$$
\begin{tikzcd}
\mathbb{Q} \arrow[dr, bend right] & & \mathbb{Q} \times 2^\omega \arrow[dl, bend left] \\
& \omega^\omega \arrow[d] & \\
& \mathbb{Q} \times \omega^\omega \arrow[dl, bend right] \arrow[dr, bend left] & \\
\mathbb{T} \arrow[d] & & \mathbb{S} \arrow[d] \\
\mathbb{Q} \times \mathbb{T} \arrow[dr, bend right] & & \mathbb{Q} \times \mathbb{S} \arrow[dl, bend left] \\
& \dots &
\end{tikzcd}
$$

The definition of filter

Definition (Wikipedia, 2016)

A *filter* is a coffee-brewing utensil, usually made of disposable paper.

This enables it to trap the coffee grounds and allow the liquid coffee to flow through.

The definition of filter (for real)

Whenever $\mathcal{X} \subseteq \mathcal{P}(\omega)$, we will identify $\mathcal{X}$ with the subspace of 2^ω consisting of the characteristic functions of elements of $\mathcal{X}$.

Definition

A *semifilter* is a collection $\mathcal{S} \subseteq \mathcal{P}(\omega)$ that satisfies the following conditions:

1. $\varnothing \notin \mathcal{S}$ and $\omega \in \mathcal{S}$
2. If $X \in \mathcal{S}$ and $X =^* Y \subseteq \omega$ then $Y \in \mathcal{S}$
3. If $X \in \mathcal{S}$ and $X \subseteq Y \subseteq \omega$ then $Y \in \mathcal{S}$

Notice that $\mathrm{Fin} \cap \mathcal{S} = \varnothing$ and $\mathrm{Cof} \subseteq \mathcal{S}$ for every semifilter $\mathcal{S}$. In particular, no semifilter is locally compact.

Definition

A *filter* is a semifilter $\mathcal{F}$ such that the following holds:

4. If $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$

Filters are deliciously homogeneous!

$$
\begin{tikzcd}
 & \text{Filter} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Topological group} \arrow[dr] & \text{Semifilter} \arrow[d] & \text{h-Homogeneous} \arrow[dl] \\
 & \text{Homogeneous} &
\end{tikzcd}
$$

A characterization of Borel filters

As we have seen, the combinatorial structure of filters imposes strong constraints on their topology. But is it possible to go in the other direction as well?
In other words, given a space, is it possible to recognize whether it is homeomorphic to a filter?
This problem has a very elegant solution in the Borel realm:

Theorem (van Engelen, 1994)

Let X be a zero-dimensional Borel space that is not locally compact. Then the following conditions are equivalent:

- *X is homeomorphic to a filter*
- *X is homogeneous, meager, and homeomorphic to its square*

The above characterization inspired the following $\mathsf{ZF} + \mathsf{DC}$ result:

Theorem (Medini and Zdomskyy, 2016)

Every filter is homeomorphic to its square.

What about semifilters?

Theorem (Medini, 2019)

Let X be a zero-dimensional Borel space that is not locally compact. Then the following conditions are equivalent:

- *X is homeomorphic to a semifilter*
- *X is homogeneous*

Easy counterexamples show that the "Borel" assumption cannot be altogether dropped in ZFC, but the following two natural questions are open (hopefully, not for long):

Question

Under AD, can the "Borel" assumption be dropped in the above characterization of semifilters?

Question

Under AD, can the "Borel" assumption be dropped in van Engelen's characterization of filters?

Two concrete non-trivial examples: $\mathbb{S}$ and $\mathbb{T}$

Theorem (van Mill, 1983; van Douwen)

Let X be a zero-dimensional space.

- $X \approx \mathbb{S}$ *if and only if X is the union of a complete subspace and a σ-compact subspace, X is nowhere σ-compact, and X is nowhere the union of a complete and a countable subspace*
- $X \approx \mathbb{T}$ *if and only if X is the union of a complete subspace and a countable subspace, X is nowhere σ-compact, and X is nowhere complete*

Fix infinite sets Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = \omega$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Define

$$\mathcal{T} = \{x_1 \cup x_2 : x_1 \subseteq \Omega_1,\ x_2 \subseteq \Omega_2, \text{ and}$$

$$(x_1 \notin \mathrm{Fin}(\Omega_1) \text{ or } x_2 \in \mathrm{Cof}(\Omega_2))\}$$

It is clear that $\mathcal{T}$ is a semifilter. Furthermore, $\mathcal{T}$ is the union of the following spaces:

- $\{x \subseteq \omega : x \cap \Omega_1 \notin \mathrm{Fin}(\Omega_1)\} \approx \omega^\omega \times 2^\omega \approx \omega^\omega$
- $\{x_1 \cup x_2 : x_1 \in \mathrm{Fin}(\Omega_1) \text{ and } x_2 \in \mathrm{Cof}(\Omega_2)\} \approx \mathbb{Q}$

Using the fact that $\mathcal{T}$ is homogeneous, one can easily see that $\mathcal{T}$ is nowhere σ-compact and nowhere complete. Hence $\mathcal{T} \approx \mathbb{T}$.

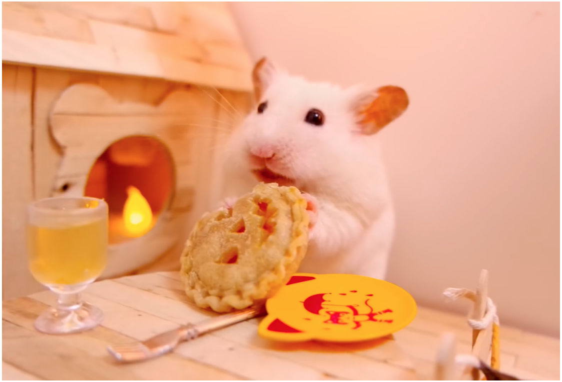
To describe $\mathbb{S}$, also fix an infinite $\Omega \subseteq \Omega_2$ such that $\Omega_2 \setminus \Omega$ is infinite. Define

$$\mathcal{S} = \{x_1 \cup x_2 : x_1 \subseteq \Omega_1,\ x_2 \subseteq \Omega_2, \text{ and}$$

$$(x_1 \notin \mathrm{Fin}(\Omega_1) \text{ or } \Omega \subseteq^* x_2)\}$$

Using an argument similar to the one that works for $\mathcal{T}$, one can show that $\mathcal{S} \approx \mathbb{S}$.

Thank you for your attention



and have a good evening!